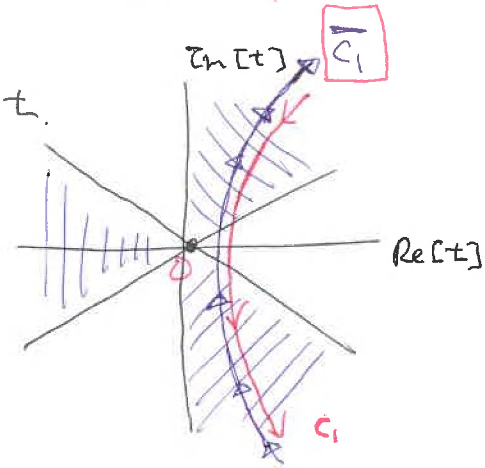
② $z \to -\infty$: Let $\zeta = -|z|^{\frac{1}{2}} t$ (inversion w.r.t. O)

$$Ai(z) = -\frac{1}{2\pi i} |z|^{\frac{1}{2}} \int_{C_1} e^{|z|^{\frac{3}{2}}\left(t + \frac{t^3}{3}\right)} dt$$

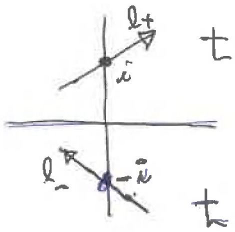
$$= \frac{1}{2\pi i} |z|^{\frac{1}{2}} \int_{\bar{C}_1} e^{|z|^{\frac{3}{2}}\left(t + \frac{t^3}{3}\right)} dt$$



→ Saddle point approx.

(two points : $t = \pm i$
equivalent

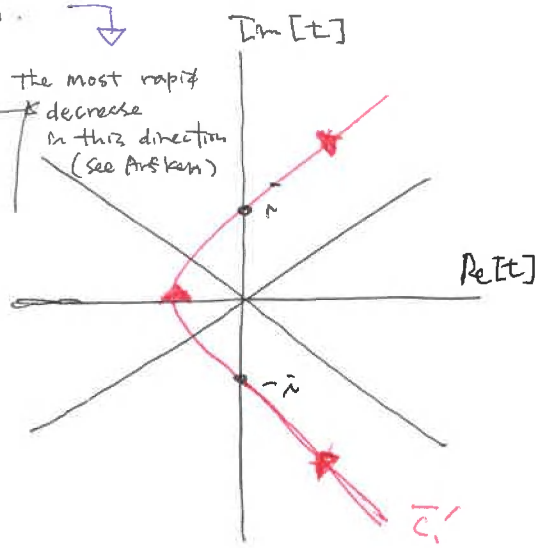
→ Deform $\bar{C}_1$ to cross $t = \pm i$.



$t = i + \zeta\sqrt{i}$ $\quad \| \sqrt{i} = e^{\frac{\pi}{4}i} = \frac{1+i}{\sqrt{2}}$

$t = -i - \zeta\sqrt{-i}$ $\quad \| \sqrt{-i} = e^{-\frac{\pi}{4}i} = \frac{1-i}{\sqrt{2}}$

The most rapid decrease in this direction (see Arfken)



$$t + \frac{t^3}{3} \simeq \frac{2}{3}i - \zeta^2 \quad \text{around } t = +i$$

$$\simeq -\frac{2}{3}i - \zeta^2 \quad \text{around } t = -i$$

$$Ai(z) \simeq \frac{1}{2\pi i} |z|^{\frac{1}{2}} \left[e^{\frac{\pi}{4}i} \int_{l_+} d\zeta \, e^{|z|^{\frac{3}{2}}\left(\frac{2}{3}i - \zeta^2\right)} - e^{-\frac{\pi}{4}i} \int_{l_-} d\zeta \, e^{|z|^{\frac{3}{2}}\left(-\frac{2}{3}i - \zeta^2\right)} \right]$$

approx. $\int_{l_\pm} d\zeta \to \int_{-\infty}^{\infty} d\zeta$ (integrand is only important near $\zeta = 0$)

$$\simeq \frac{1}{2\sqrt{\pi}\, i} |z|^{-\frac{1}{4}} \left[e^{i\left(\frac{2}{3}|z|^{\frac{3}{2}} + \frac{\pi}{4}\right)} - e^{-i\left(\frac{2}{3}|z|^{\frac{3}{2}} + \frac{\pi}{4}\right)} \right]$$

$$= \frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}} \sin\left[\frac{2}{3}|z|^{\frac{3}{2}} + \frac{\pi}{4}\right]$$

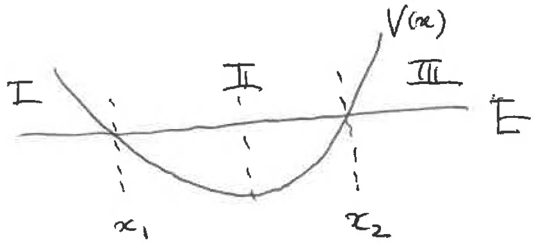
or

$$Ai(z) \simeq \frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}} \cos\left[\frac{2}{3}|z|^{\frac{3}{2}} - \frac{\pi}{4}\right] \quad \text{as } z \to -\infty$$

Similarly,

$$Bi(z) \simeq \begin{cases} \frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}} e^{\frac{2}{3}|z|^{\frac{3}{2}}} & \text{as } z \to \infty \quad \leftarrow \text{unphysical!} \\ -\frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}} \sin\left(\frac{2}{3}|z|^{\frac{3}{2}} - \frac{\pi}{4}\right) & \text{as } z \to -\infty \end{cases}$$

Now, coming back to



I : $u_E(x) \sim \exp\left[-\frac{1}{\hbar}\int_x^{x_1} dx' \sqrt{2m[V(x')-E]}\right]$

near x_1, $V(x) \simeq E + |V'(x_1)|(x_1 - x) + \cancel{\cdots}$

$$\Rightarrow \frac{1}{\hbar}\int_x^{x_1} dx' \sqrt{2m(x_1-x')|V'(x_1)|} = \sqrt{\frac{2mV'}{\hbar^2}} \cdot \frac{2}{3}(x_1-x)^{\frac{3}{2}}$$

$$= \frac{2}{3} z^{\frac{3}{2}} \quad \Big\| \quad z = \left(\frac{2mV'}{\hbar^2}\right)^{\frac{1}{3}}(x_1 - x)$$

$$\therefore u_E(x) \sim Ai(z) \quad \text{as } z \to \infty$$

III : $u_E(x) \sim \exp\left[-\frac{1}{\hbar}\int_{x_2}^{x} dx' \sqrt{2m[V(x')-E]}\right]$

near x_2, $V(x) \simeq E + |V'(x_2)|(x - x_2) + \cancel{\cdots}$

$$\Rightarrow \frac{1}{\hbar}\int_{x_2}^{x} dx' \sqrt{2m|V'(x_2)|(x'-x_2)} = \sqrt{\frac{2mV'}{\hbar^2}} \frac{2}{3}(x-x_2)^{\frac{3}{2}}$$

$$= \frac{2}{3} z^{\frac{3}{2}} \quad \Big\| \quad z = \left(\frac{2mV'}{\hbar^2}\right)^{\frac{1}{3}}(x - x_2)$$

$$\therefore u_E(x) \sim Ai(z) \quad \text{as } z \to \infty$$

II : ① from x_1

$$u_E(x) \sim \cos\left[\frac{1}{\hbar}\int_{x_1}^{x} dx' \sqrt{2m(E-V(x'))} - \frac{\pi}{4}\right]$$

def. of z in I $\rightarrow$ $\frac{2}{3}|z|^{\frac{3}{2}}$

from $Ai(z)$ as $z \to -\infty$.

② from x_2

$$u_E(x) \sim \cos\left[\frac{1}{\hbar}\int_{x}^{x_2} dx' \sqrt{2m[E-V(x')]} - \frac{\pi}{4}\right].$$

def. of z in III $\rightarrow$ $\frac{2}{3}|z|^{\frac{3}{2}}$

from $Ai(z)$ as $z \to -\infty$.

Phase Matching For ALL x.

$$\frac{1}{\hbar}\int_{x_1}^{x} dx' [\quad] - \frac{\pi}{4} = -\frac{1}{\hbar}\int_{x}^{x_2} dx' [\quad] + \frac{\pi}{4} + n\pi$$

$\Leftarrow \cos\theta = \cos(-\theta)$.

$\Rightarrow$ Quantization condition. (WKB)

$$\int_{x_1}^{x_2} dx' \sqrt{2m[E-V(x')]} = \left(n+\frac{1}{2}\right)\pi\hbar$$

$\Leftrightarrow$ Sommerfeld – Wilson quantization (old Quantum theory)

$$\oint p\, dq = nh.$$

$$\| \quad p_{cl} = \sqrt{2m(E-V)}$$

• Why is it "semi-classical"?

(MORE RIGOROUSLY....)

– Hamilton-Jacobi Theory

Classical Action $S(b,a) = \int_{t_a}^{t_b} dt\, L(q(t), \dot{q}(t), t)$.

where $b \equiv q_b \equiv (q_1(t_b), q_2(t_b), \cdots q_f(t_b))$

$a \equiv q_a \equiv (q_1(t_a), q_2(t_a), \cdots q_f(t_a))$.

def. Integral of L along the trajectory allowed by the eq. of motion from configuration q_a at t_a to q_b at t_b.

It's a physical trajectory.

Taking an arbitrary variation δq,

$$\delta S(b,a) = \int_{t_a}^{t_b} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i$$

$= 0$: We're on a physical trajectory!

* NOTE ! $\int_t \frac{\partial L}{\partial \dot{q}} \delta \dot{q}\, dt$ ↳ int. by parts $\frac{\partial L}{\partial \dot{q}} \dot{q} \Big|_t - \int_t \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \delta q\, dt$

$$+ \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{t_a}^{t_b}.$$

$\neq 0$: We do have variations at the end points!

$$\Rightarrow \delta S(b,a) = P_{b,i}\, \delta q_i(t_b) - P_{a,i}\, \delta q_i(t_a).$$

$$\Rightarrow P_{b,i} = \frac{\partial}{\partial q_{b,i}} S(b,a) \equiv P_{b,i}(b;a), \quad P_{a,i} = -\frac{\partial}{\partial q_{a,i}} S(b,a) \equiv P_{a,i}(b,a)$$

Fix (q_a, t_a),

Consider the change of action in t_b

$$S(b,a) = \int_{t_a}^{t_b} dt\, L(q, \dot{q}; t)$$

$$\Rightarrow \quad \frac{dS}{dt_b} = L(t_b)$$

$$\hookrightarrow \quad = \frac{\partial S}{\partial t_b} + \frac{\partial S}{\partial q_{b,i}} \dot{q}_{b,i} = \frac{\partial S}{\partial t_b} + P_{b,i} \dot{q}_{b,i}$$

Thus,
$$\frac{\partial S}{\partial t_b} + \underbrace{P_{b,i} \dot{q}_{b,i} - L(t_b)}_{\equiv H(t_b)} = 0$$

$t \equiv t_b$
$q = q_b$

Then. $S \equiv S(q_1 \cdots q_f, t)$

for $H \equiv H\left(q_1 \cdots q_f, \frac{\partial S}{\partial q_1} \cdots \frac{\partial S}{\partial q_f}; t\right)$.

$$\Rightarrow \quad \boxed{\frac{\partial S}{\partial t} + H = 0} \Rightarrow \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) = 0 \; //$$

Hamilton - Jacobi equation

- For a time-indep. Hamiltonian, $(H = E)$

$$S(q, t) = W(q) - Et \quad \left(\text{integrate } \frac{\partial S}{\partial t} + H = 0\right)$$

Hamilton's characteristic function.

- If we consider a free particle in a potential

$$|\nabla S| = |\nabla W| = \sqrt{2m(E-V)}$$

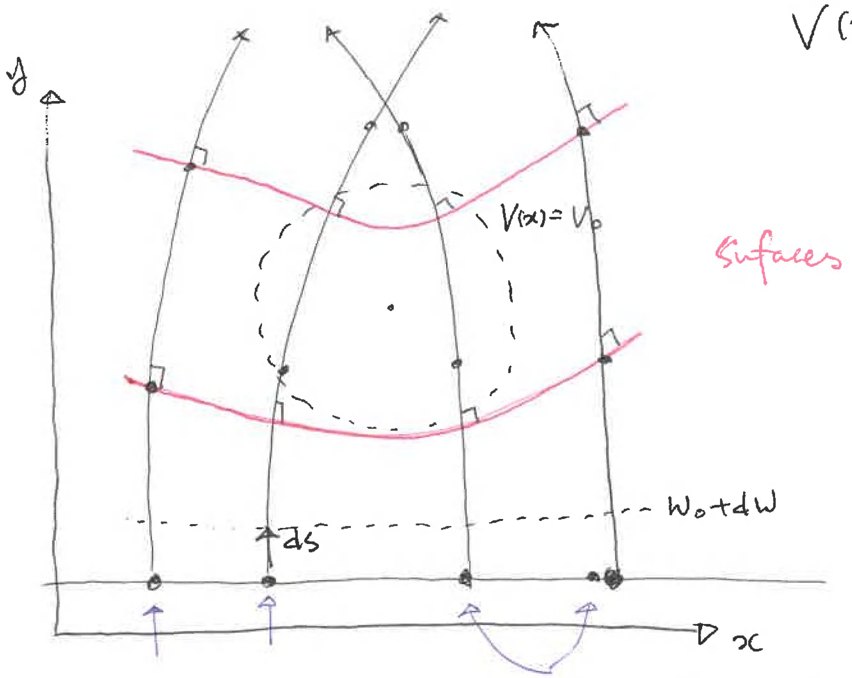
$=$ momentum. //

1D : If $V = 0$.
$W(x) \sim px$
$S(x,t) \sim px - Et$.

※ Meaning of S and W.

Consider a particle moving in a potential

$$V(x) = -\frac{1}{1+x^2+y^2}$$



Surfaces of $W(q) = \text{const}$
(equi-W planes)

Along the trajectory, length ds for $W_0 \to W_0 + dW$

$$\Rightarrow \quad ds = \frac{dW}{|\nabla W|} = \frac{dW}{\sqrt{2m[E-V_q]}} \qquad [\ V_q \equiv V(q)]$$

The motion of $S(q,t) = \text{constant}$, $\left(dW = E\,dt \right)$ $\because dS = 0$.

$$\Rightarrow \quad \frac{ds}{dt} = \frac{E}{\sqrt{2m[E-V_q]}}$$

It's like $S \sim px - Et$.

If the surfaces of $S = \text{const.}$ as wave-fronts,

[ex. $e^{i(kx-\omega t)}$ (ϕ)] $\to v = \left.\frac{dx}{dt}\right|_\phi = \frac{\omega}{k}$

$$\frac{ds}{dt} = \text{phase velocity}.$$

$\Rightarrow$ Classical Mechanics (particles are not on the surface (wave fronts))

$\approx$ Geometrical optics.

→ Semi classical Interpretation of the Wave function (Q.M.)

- From Hamilton-Jacobi Theory (C.M.)

$S \doteq$ phase factor of wave fn.

- Similarity to WKB approx.

HJ eq. for $H = \frac{p^2}{2m} + V(x)$ (t-indep.)

$$\frac{\partial S}{\partial t} + H = 0 \quad \text{with} \quad S = W(x) - Et$$

$$\Rightarrow \quad -E + \frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + V(x) = 0$$

... HJ eq.

$$\hookrightarrow \quad -\left(\frac{\partial W}{\partial x}\right)^2 + \boxed{\frac{2M}{\hbar^2}\left(E - V(x)\right)} \cdot \hbar^2 = 0 \quad \text{(C.M.)}$$

→ This is just WKB!

where $u_E(x) = e^{iW(x)/\hbar}$

Q.M.

⟹ Brillouin-Wentzel Ansatz

$$\Psi(x,t) = \exp\left[i\,\Theta(x,t)/\hbar\right]$$

$\Theta \to S$

Schrodinger eq.

⟶

$\hbar$ is here only.

$$\frac{\partial \Theta}{\partial t} + \frac{1}{2m}\left(\frac{\partial \Theta}{\partial x}\right)^2 + V(x) = \frac{i\hbar}{2m} \cdot \frac{\partial^2 \Theta}{\partial x^2}$$

$\boxed{\left(\frac{\partial S}{\partial x}\right)^2 \gg \hbar\left|\frac{\partial^2 S}{\partial x^2}\right|}$

as in WKB.

Without this term, the Schrodinger eq. is just the HJ eq.

Thus, we may just write down

$$\Psi \sim \exp\left[\frac{i}{\hbar} \cdot \text{Classical Action}\right] \quad \text{when } \hbar \to 0.$$

To see the quantum corrections, consider

$$\Theta = S + \frac{\hbar}{i} S_1 + \left(\frac{\hbar}{i}\right)^2 S_2 + \cdots$$

$\uparrow$ classical.

$\Rightarrow$ $O(\hbar^0)$: $$\underbrace{\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V(x) = 0}_{\text{H-J eq.}}$$

$O(\hbar^1)$: $$\frac{\partial S_1}{\partial t} + \frac{1}{2m}\left[2 \cdot \frac{\partial S}{\partial x} \cdot \frac{\partial S_1}{\partial x} + \frac{\partial^2 S}{\partial x^2}\right] = 0.$$

$\to$ Find S_1 for a give S.

$\Rightarrow$ Semi classical wave function

$$\Psi_{sc}(x,t) = \sqrt{\rho(x,t)}\; e^{\frac{i S(x,t)}{\hbar}}$$

where $$\rho(x,t) = |\Psi_{sc}|^2 = e^{2S_1(x,t)}$$

$\nearrow$ prob. density of finding a particle near x at t.

(Max Born)

* Remark

"classical" means here when $\Theta \to S$.

for a very small S_1, HJ eq. is recovered when

$$\left(\frac{\partial S}{\partial x}\right)^2 \gg \hbar\left(\frac{\partial^2 S}{\partial x^2}\right).$$

; short wave length

(See WKB.)

Rewriting $O(\hbar^1)$ terms for $\rho = e^{2S_1}$,

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x}\left[\frac{1}{m}\cdot\frac{\partial S}{\partial x}\cdot\rho(x,t)\right] = 0.$$

using the current density $\vec{J}(\vec{x},t)$

$$\vec{J}(\vec{x},t) = \frac{\hbar}{m}\,\mathrm{Im}\,[\underbrace{\Psi^*\nabla\Psi}_{\leftrightarrow\sqrt{\rho}\nabla\sqrt{\rho}+\frac{i}{\hbar}\rho\nabla S}] = \frac{\rho\cdot\nabla S}{m}.$$

$\Downarrow$

$\int d^3\vec{J} = \frac{\langle\vec{p}\rangle}{m}$ (a single particle!)

Thus. $O(\hbar^1)$ terms give.

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0.$$: continuity eq.

: prob. flux is conserved! It just behaves like a "flux".

You can directly show this from the Schrödinger eq, also.

Hamilton-Jacobi equation particle

particle - wave duality

as $\hbar \to 0$, short wavelength

wave "Geometrical optics"

conceptually ...

Schrödinger equation wave